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On the matching method and the Goldstone theorem in holography

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Abstract

We study the transition of a scalar field in a fixed AdS_{d+1} background between an extremum and a minimum of a potential. We first prove that the potential involved cannot be generic, i.e. that a fine-tuning of their parameters is mandatory for the solution to exist. We compute analytically the solution to the perturbation equation by generalizing the usual matching method to higher orders and find the propagator of the boundary theory operator defined through the AdS-CFT correspondence. We show that it always presents a simple pole at $q^2 = 0$ in accordance with the Goldstone theorem applied to a spontaneously broken dilatation invariance. The result is supported also by a WKB calculation.

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Contents

1 Introduction and results

The gravity plus real scalar field system in asymptotic anti de Sitter (AdS) spaces is the simplest playground for the AdS-CFT correspondence [1, 2, 3]. It is relatively easy to solve and it gives some insight into what the correspondence is and what it means (for a partial list see [4, 5, 6, 7, 8, 9, 10]). Unfortunately several simple statements get a bit obscured by technical details and the complexity of the gravity system. Since one would expect that weakly coupled gravity has a smooth non-interacting limit[4], it may be useful to study the no back-reaction limit of this system: a real scalar field in a fixed AdS background, and no gravity at all (for some reviews which treat the subject see for example [12, 13, 14, 15]). Recently this has been done in [16], where the potential of the real scalar field presenting a UV extremum and a IR minimum has been approximated by a piece-wise quadratic potential in order to allow analytic treatment. It has been then shown

[4] In particular cases there could be subtle exceptions though, see for example [11].

that: a) in order for the solution between the extremum and the minimum to exist, there must be some non-trivial constraint among parameters in the potential; b) such a solution has vanishing action; c) at least one region needs $V'' < -d^2/4$ and d) the propagator in the boundary theory exhibits a simple $1/q^2$ pole as predicted by the Goldstone theorem applied to the spontaneous breaking of the dilatation invariance [17]. Conclusions a), b) and c) were not completely understood while d), although expected on general grounds, has been recently called into question in [18], where a $1/q^{2\nu_{IR}}$ propagator with $\nu_{IR} > d/2$ has been found in the $q \to 0$ limit. The purpose of this paper is to generalize the piece-wise quadratic potential of [16] to more general ones in order to shed light on conclusions of our previous paper, with the main goal to clarify which is the right low energy behavior of the propagator, the (expected on general grounds) $1/q^2$ of [17, 16] or the (surprising) $1/q^{2\nu_{IR}}$ of [18].

In particular, we show that a consistent treatment of the matching method requires a next-to-leading order calculation. The result confirms what we found in [16]: the propagator of the boundary theory defined through the AdS-CFT correspondence in the gravity no-backreaction limit has a simple pole at $q^2 = 0$, signaling the presence of a Goldstone mode. The argument can be summarized as follows. The leading order solution for the perturbation in a small q expansion is given by

$$\xi(z;q) \approx C_+(q)\ \xi_+(z) + C_-(q)\ \xi_-(z) \tag{1.1}$$

with the limits,

$$\xi_\pm(z) \xrightarrow{z\to 0} a_\pm^{UV}\ z^{\Delta_\pm^{UV}} \qquad ; \qquad \xi_\pm(z) \xrightarrow{z\to\infty} a_\pm^{IR}\ z^{d-\Delta_\pm^{IR}} \tag{1.2}$$

There is only one relevant correction to the leading order solution for $q \to 0$:

$$\xi(z;q) \approx C_+(q)\ \big(\xi_+(z) + q^2\ \delta\xi_+(z)\big) + C_-(q)\ \xi_-(z) \tag{1.3}$$

This next-to-leading order correction $\delta\xi_+(z)$ behaves in the limiting cases as [5]

$$\delta\xi_+(z) \xrightarrow{z\to 0/\infty} \epsilon_{++}^{UV/IR}\ \xi_-(z) \tag{1.4}$$

The matching with the (conveniently normalized) Bessel K solution for $z \to \infty$ is done at fixed, small $q\,z$:

$$C_+(q)\ a_+^{IR}\ z^{\Delta_-^{IR}} + \big(C_-(q) + q^2\ C_+(q)\ \epsilon_{++}^{IR}\big)\ a_-^{IR}\ z^{\Delta_+^{IR}} \approx z^{\Delta_-^{IR}} + \gamma\ q^{2\nu_{IR}}\ z^{\Delta_+^{IR}} \tag{1.5}$$

with the result for $\nu_{IR} > 1$ (always true for $d > 2$)

$$C_+(q) = 1/a_+^{IR} \qquad , \qquad C_-(q) = -C_+(q)\ \epsilon_{++}^{IR}\ q^2 \tag{1.6}$$

[5] We keep here the same apparently obscure notation as in the body of the paper: it will become clear later on.

On the other side the boundary propagator is found according to the AdS/CFT prescription at the opposite limit $z \to 0$:

$$C_+(q)\, a_+^{UV}\, z^{\Delta_+^{UV}} + \left(C_-(q) + q^2\, C_+(q)\, \epsilon_{++}^{UV}\right)\, a_-^{UV}\, z^{\Delta_-^{UV}} \equiv \frac{a_+^{UV}}{a_+^{IR}} \left(z^{\Delta_+^{UV}} + \frac{z^{\Delta_-^{UV}}}{G_2(q)} \right) \tag{1.7}$$

from where for $q \to 0$

$$G_2(q) = \frac{a_+^{UV}/a_-^{UV}}{\epsilon_{++}^{UV} - \epsilon_{++}^{IR}} \times \frac{1}{q^2} \tag{1.8}$$

follows. Having we stopped at the formally leading order (i.e. without $\delta\xi_+(z)$ or ϵ_{++}^{UV}), we would get a completely different (and wrong) propagator in the $q \to 0$ limit:

$$G_2(q) \to \left(\frac{a_+^{UV}}{a_-^{UV}}\right) \left(\frac{a_-^{IR}}{a_+^{IR}}\right) \frac{1}{\gamma} \times q^{-2\nu_{IR}} \tag{1.9}$$

This is the essence of the argument, all details will be given in the paper.

Our philosophy in obtaining these results will be similar to that in [16], which has become usual in applications of the AdS/CFT duality to condensed matter systems (see for example [19]): we do not know which the dual Lagrangian on the boundary is, but we simply define the boundary theory from the bulk theory through the holographic dictionary, hoping that it gives sensible answers and is consistent with the usual quantum field theory rules.

The plan of the paper is the following. After setting the notation and main formulae in Section 2.1, we first show in Section 2.2 that even in this no-backreaction limit one can get a BPS-type solution to the first order equation of motion provided that the potential is written in terms of a properly defined superpotential. In Sections 2.2.1, 2.2.2 and 2.2.3 we give some examples of analytically non-trivial solutions. Then in Section 2.3 we prove in complete generality that for the solution to exist, the second derivative of the scalar potential must be smaller than $-d^2/4$ in at least some region. In Section 2.4 we explain why the potential cannot be arbitrary, i.e. a relation among parameters is mandatory. This clarifies the results in [16], which were found for piece-wise quadratic potentials only and have not been understood. The most important part of the paper is Section 3, where we describe three methods for the calculation of the boundary propagator: in 3.1 we briefly summarize [16], in 3.2 we describe in details the matching method, while in 3.3 we use the WKB approximation. Finally, a long Appendix A is devoted to a detailed analysis of the matching method to all orders.

2 Preliminaries

2.1 The system

We consider a real scalar field ϕ in $d+1$ dimensions with bulk euclidean action

$$S^{(bulk)}[\phi] = \int d^{d+1}x\, \sqrt{\det g_{ab}}\, \left(\frac{1}{2} g^{ab}\, \partial_a\phi\, \partial_b\phi + U(\phi) \right) \tag{2.1}$$

in a non-dynamical AdS_{d+1} background

$$g = \frac{1}{z^2}\left(L^2 dz^2 + \delta_{\mu\nu}\, dx^\mu\, dx^\nu\right) \tag{2.2}$$

where (x^μ) are the QFT coordinates with $x^d \equiv i\, x^0$ the euclidean time, and L the AdS scale. The boundary is located at $z = 0$ (UV region) while the horizon is at $z = \infty$ (IR region).

The equation of motion derived from (2.1) results,

$$z^2\, \ddot{\phi}(x,z) - (d-1)\, z\, \dot{\phi}(x,z) + L^2\, z^2\, \Box\phi(x,z) = L^2\, U'(\phi) \tag{2.3}$$

where $\Box \equiv \delta_{\mu\nu}\partial_\mu\partial_\nu$, and throughout the paper we will indicate with a dot the derivative w.r.t. the bulk coordinate z and with a prime a field derivative.

We will consider potentials with

$$U(0) = 0 \quad , \quad U'(0) = 0 \qquad ; \qquad U(\phi_m) < 0 \quad , \quad U'(\phi_m) = 0 \quad , \quad U''(\phi_m) > 0 \tag{2.4}$$

i.e. ϕ_m will be the true minimum, while at the origin the potential can have a minimum (being a false vacuum thus) or even a maximum, provided that it is in the Breitenlöhner-Freedman conformal window $-d^2/4 < L^2\, U''(0) < 0$.

It will be useful along the paper to work with dimensionless field variable and potential defined by,

$$t(z,x) \equiv \frac{\phi(z,x)}{\phi_m} \qquad ; \qquad V(t) \equiv \frac{L^2}{\phi_m^2}\, U(\phi_m\, t) \tag{2.5}$$

We will be interested in regular, Poincarè invariant solutions $t = t(z)$ that interpolate between the UV and IR regions. They obey the equation of motion

$$z^2\, \ddot{t}(z) - (d-1)\, z\, \dot{t}(z) = V'(t) \tag{2.6}$$

and necessary behave in the UV and IR as

$$t(z) \xrightarrow{z\to 0} a_{UV}\, z^{\Delta^{UV}} \qquad ; \qquad t(z) \xrightarrow{z\to\infty} 1 + a_{IR}\, z^{d-\Delta^{IR}} \tag{2.7}$$

respectively, where

$$\Delta^{UV/IR} \equiv \frac{d}{2} + \nu_{UV/IR} \qquad ; \qquad \nu_{UV/IR} \equiv \sqrt{\frac{d^2}{4} + m^2_{UV/IR}} \tag{2.8}$$

with $m^2_{UV} \equiv V''(0)$ and $m^2_{IR} \equiv V''(1) > 0$ ($t = 1$ is a minimum according to (2.4))[6].

We recall as a last remark that the symmetries of AdS space translate in the scale invariance of equation (2.6), i.e. if $t(z)$ is a solution so it is $t(\lambda z)$, a fact of great relevance in what follows.

[6] In the window $-\frac{d^2}{4} < V''(0) < 0$ the term $z^{d-\Delta^{UV}}$ could also be present in the small z power expansion of $t(z)$. From the AdS/CFT point of view this term is interpreted as a source that breaks explicitly the scale invariance of the boundary QFT; then we should not expect a Goldstone mode to appear, situation we are not interested in. These domain walls are interpreted as dual to renormalization group flows generated by deformation of the UV CFT by a relevant operator, i.e. one of dimension less than d [15].

2.2 The BPS solutions

The fact that the on-shell action vanishes[7] [16] is a hint that the solution may be of the BPS type, i.e. it solves a first order equation. Let us prove this statement in our context. The action (2.1) is $S^{(bulk)}[\phi] = \frac{V_d \, \phi_m{}^2}{L} \, I[t]$, with

$$I[t] = \int_0^\infty dz \; z^{-1-d} \, \left(\frac{1}{2} \, z^2 \; \dot{t}(z)^2 + V(t) \right) \tag{2.9}$$

If we define the "superpotential" $W(t)$ by,

$$V(t) = \frac{1}{2} \, W'^2(t) - d \, W(t) \tag{2.10}$$

then

$$\begin{aligned} I[t]|_{on-shell} &= \int_0^\infty dz \left[z^{-1-d} \frac{1}{2} \; \left(z \; \dot{t}(z) - W'(t) \right)^2 + z^{-d} \; \dot{t}(z) \; W'(t) - d \; z^{-1-d} \; W(t) \right] \\ &= \int_0^\infty dz \; \left[z^{-1-d} \; \frac{1}{2} \; \left(z \; \dot{t}(z) - W'(t) \right)^2 + z^{-d} \; \frac{dW(t)}{dz} + \frac{d}{dz} \left(z^{-d} \right) \; W(t) \right] \\ &= \int_0^\infty dz \; z^{-1-d} \; \frac{1}{2} \; \left(z \; \dot{t}(z) - W'(t) \right)^2 \end{aligned} \tag{2.11}$$

In the last line we used,

$$\left. \frac{W(t)}{z^d} \right|_{z=0}^{z=\infty} = - \left. \frac{W(t)}{z^d} \right|_{z=0} = 0 \tag{2.12}$$

that follows from the fact that according to (2.7) $t(z) \sim z^{\Delta^{UV}}$ for $z \to 0$, and since $V(0) = V'(0) = 0$ and $V''(0)$ finite, $W(t) \sim t^{2+n} \sim z^{n\Delta^{UV} + 2\nu_{UV} + d}$, with $n \geq 0$.

It is now obvious from (2.11) that there is a BPS like equation,

$$z \; \dot{t}(z) = W'(t(z)) \tag{2.13}$$

whose solutions satisfy also the full second order equation of motion (2.6) and for which the action (2.11) vanishes.

Before analyzing some explicit examples we would like to notice the following relevant fact. In the presence of dynamical gravity one must consider the gravity action

$$S_{grav}[g] = \frac{1}{16 \, \pi \, G_N} \, \int d^{d+1}x \, \left(R[g] + \frac{d \, (d-1)}{L^2} \right) \tag{2.14}$$

where G_N is the $d+1$-dimensional Newton constant. A Poincarè consistent ansatz that replaces (2.2) is,

$$g = \frac{1}{z^2} \left(L^2 \, \frac{dz^2}{F(z)} + \delta_{\mu\nu} \; dx^\mu \; dx^\nu \right) \tag{2.15}$$

[7] This could have been expected from the spontaneous breaking of conformal symmetry, for a discussion on this point and references see for example [20].

If we now introduce the superpotential by replacing (2.10) with,

$$V(t) \equiv \frac{1}{2} W'^2(t) - d\, W(t) - \frac{d\,\kappa^2}{4}\, W^2(t) \tag{2.16}$$

where $\kappa^2 \equiv \frac{8\,\pi}{d-1} {\phi_m}^2\, G_N$, then (2.13)gets replaced by the following equations of motion,

$$\begin{aligned} F(z) &= \left(1 + \frac{\kappa^2}{2}\, W(t)\right)^2 && (2.17) \\ z\, \dot{t}(z) &= \frac{W'(t(z))}{1 + \frac{\kappa^2}{2}\, W(t)} && (2.18) \end{aligned}$$

So, we conclude that there exist a smooth non-dynamical gravity $\kappa \to 0$ limit which yields the system under consideration.

2.2.1 The simplest nontrivial example

The simplest consistent nontrivial solution is for a quartic potential, i.e. a cubic "superpotential" W. The choice

$$W(t) = \Delta\, \left(\frac{1}{2}\, t^2 - \frac{1}{3}\, t^3\right) \qquad ; \qquad \Delta > \frac{d}{2} \tag{2.19}$$

has the right properties (2.4), i.e. $V(0) = V'(0) = V'(1) = 0\,,\, V(1) = -\frac{d\,\Delta}{6} < 0\,,\, V''(1) = \Delta\,(\Delta + d) > 0$. Furthermore $V''(0) = \Delta\,(\Delta - d)$ which implies that $t = 0$ is a minimum when $\Delta > d$ and a maximum if $0 < \Delta < d$. The solution to (2.13) is,

$$t(z) = \frac{z^\Delta}{1 + z^\Delta} \tag{2.20}$$

where we have fixed the scale invariance freedom. On the other hand, the parameter Δ must be identified with Δ^{UV} in (2.8) (or with $d - \Delta^{UV}$ if $0 < \Delta < \frac{d}{2}$, but we will not consider this case, see footnote on the previous page) while that in the IR region $z \to \infty$ the solution goes like $1 - z^{d - \Delta^{IR}}$ with

$$\Delta^{IR} = d + \Delta \tag{2.21}$$

2.2.2 The Z_2 symmetric case

Another interesting case consists of the sixth order potential with the Z_2 symmetry $t \to -t$. The ansatz for the superpotential

$$W(t) = \Delta\, \left(\frac{1}{2}\, t^2 - \frac{1}{4}\, t^4\right) \tag{2.22}$$

leads to the solution,

$$t(z) = \frac{z^\Delta}{(1 + z^{2\Delta})^{1/2}} \tag{2.23}$$

Similar remarks as before applies in what regards the relation between Δ^{UV} and the parameter Δ, but in the infrared now we find,

$$\Delta^{IR} = d + 2\,\Delta \tag{2.24}$$

2.2.3 A case with Δ^{UV} and Δ^{IR} independent

In the above examples a correlation between the UV and IR Δ's is present. This is however not a generic feature of the system. In fact, choosing for example

$$W(t) = -\frac{1}{4}\,\left(\Delta^{IR} - (d + \Delta^{UV})\right)\,t^4 + \frac{1}{3}\,\left(\Delta^{IR} - (d + \Delta^{UV}) - \Delta^{UV}\right)\,t^3 + \frac{\Delta^{UV}}{2}\,t^2 \tag{2.25}$$

we get the solution

$$z(t) = \left[\frac{\Delta^{UV} + \left(\Delta^{IR} - (d + \Delta^{UV})\right)\,t}{1 - t}\right]^{\frac{1}{\Delta^{IR}-d}} \left[\frac{t}{\Delta^{UV} + (\Delta^{IR} - (d + \Delta^{UV}))\,t}\right]^{\frac{1}{\Delta^{UV}}} \tag{2.26}$$

which has the limits (2.7) with

$$a_{UV} = \left(\Delta^{UV}\right)^{\frac{\Delta^{IR}-\left(d+\Delta^{UV}\right)}{\Delta^{IR}-d}} \qquad ; \qquad a_{IR} = -\left(\Delta^{IR} - d\right)^{-\frac{\Delta^{IR}-\left(d+\Delta^{UV}\right)}{\Delta^{UV}}} \tag{2.27}$$

The parameters $\Delta^{UV} > d/2$ (corresponding to the maximum or minimum in the UV) and $\Delta^{IR} > d + \Delta^{UV}$ (minimum in the IR) can be otherwise arbitrary.

2.3 $V''(t) < -d^2/4$

In a piece-wise quadratic potential case it has been noticed in [16] that at least in some interval the second derivative of the potential must be smaller than $-d^2/4$ for the solution to exist. Let us here show this statement for a general potential $V(t)$ characterized by (2.10). Let us define

$$F \equiv \int d\mu\; W'(t)^2\,\left(V''(t) + \frac{d^2}{4}\right)\Bigg|_{t=t(z)} \tag{2.28}$$

where $t(z)$ is the solution of the BPS equation (2.13) and to simplify the notation we will use in this subsection the abbreviation

$$\int d\mu \cdots \equiv \int_0^\infty dz\; z^{-d-1} \ldots \tag{2.29}$$

and omit the field dependence. Our aim is to show that the quantity F is non-positive, so that $V'' < -d^2/4$ at least in some region.

First we rewrite (2.28) using (2.10)

$$F = \int d\mu\,\left(W'^2\,W''^2 + W'^3\,W''' - d\,W'^2\,W'' + \frac{d^2}{4}\,W'^2\right) \tag{2.30}$$

Now we use (assuming vanishing boundary terms, which is easily verified)

$$\int d\mu\, W'^2 = \frac{2}{d}\int d\mu\, W'^2\, W'' \tag{2.31}$$

$$\int d\mu\, W'^3\, W''' = \int d\mu \left(d\, W'^2\, W'' - 2\, W'^2\, W''^2\right) \tag{2.32}$$

to rewrite (2.30) as

$$F = \frac{d}{2}\int d\mu\, W'^2\, W'' - \int d\mu\, W'^2\, W''^2 \tag{2.33}$$

Finally we use the Schwartz inequality

$$\int d\mu\; f\; g \leq \left(\int d\mu\; f^2\right)^{\frac{1}{2}} \left(\int d\mu\; g^2\right)^{\frac{1}{2}} \tag{2.34}$$

to derive from (2.31)

$$\int d\mu\, W'^2 \leq \frac{4}{d^2}\int d\mu\, W'^2\, W''^2 \tag{2.35}$$

Using then (2.34) we get first

$$\int d\mu\, W'^2\, W'' \leq \left(\int d\mu\, W'^2\right)^{\frac{1}{2}} \left(\int d\mu\, W'^2\, W''^2\right)^{\frac{1}{2}} \tag{2.36}$$

from which finally it follows

$$F \leq 0 \tag{2.37}$$

This proves our statement: the inequality $V'' < -d^2/4$ is valid at least in some region of z for any potential V of the form (2.10).

Notice that since at the horizon ($z \to \infty$) the potential has a minimum and at the boundary ($z = 0$) a minimum or a maximum in the conformal window, there are always an even number of times that V'' crosses this particular value $-d^2/4$.

2.4 Why the potential cannot be generic

If the potential is defined through a superpotential and this last is polynomial, the potential is clearly not generic, as the explicit examples considered above show. A bit less obvious is the situation where the potential cannot be written in terms of a polynomial superpotential. As it has been noted in [16] and remarked before, the point is that the equation of motion is invariant under dilatations $z \to \lambda z$ for any positive real λ. There is thus an infinite family of solutions: the location of the domain wall is not determined.

What happens if a fine-tuned potential changes a bit, i.e. if we relax the constraint among the potential parameters? The numerical output will make $t(z)$ diverge, so that for $z \to \infty$ limit it will not reach the unit value. In other words, the transition is not from the extremum in the origin to the minimum at $t = 1$, but it escapes to infinity. In

order to make the field land to the minimum, one needs a constrained value for the model parameters.

There are two simple ways to see why there must be some constraint among the model parameters, if we are looking for a solution of (2.6).

First of all, we have a second order differential equation. In the limit $z \to 0$ this non-linear equation can be linearized, call the two independent solutions of this linearized version $t_+(z)$ and $t_-(z)$. Let they be defined so that for $z \to 0$, $t_+(z) \propto z^{\Delta^{UV}}$ with Δ^{UV} given in (2.8) and $t_-(z) \propto z^{d-\Delta^{UV}}$. This second $t_-(z)$ is interpreted in the AdS-CFT dictionary as a source. All solutions to the original full non-linear equations have to evolve only towards $t_+(z)$ for $z \to 0$ in order for the source to vanish. There is however no guarantee that these solutions are finite for $z \to \infty$. In general it will not be the case, only solutions which evolve to some linear combination $a\, t_+(z) + b\, t_-(z)$ for $z \to 0$ will be finite in the opposite limit at $z \to \infty$. We can enforce $b = 0$ and thus have a $t(z)$ sourceless at $z \to 0$ and finite at $z \to \infty$ only by carefully choosing the parameters of the original Lagrangian, i.e. the potential. From here the fine-tuning among parameters.

Another way perhaps more familiar of setting the problem is through the linearized perturbation equation around the assumed solution $t(z)$. If we write the perturbation as $\xi(z;q)\, e^{iq\cdot\frac{x}{L}}$, such equation results (A.6). We can rewrite this linearized equation for perturbations in a Schrödinger-like form. Taking $\xi(z;q) = z^{\frac{d-1}{2}}\, f(z;q)$ we get,

$$\ddot{f}(z;q) - \left[q^2 + \frac{1}{z^2}\left(\frac{d^2-1}{4} + V''(t(z))\right)\right] f(z;q) = 0 \tag{2.38}$$

Now, well-known symmetry arguments (in this case related to dilatation invariance) show that $\xi(z;0) \sim z\,\dot{t}(z)$ solves equation (A.6) with $q^2 = 0$. But (2.38) is a second order linear differential equation with two independent solutions and then standard quantum mechanics arguments work. By definition, necessary $f(z;0) \sim z^{\frac{1}{2}-\nu_{IR}}$ for $z \to \infty$ and the solution that goes as $z^{\frac{1}{2}+\nu_{IR}}$ must be discarded. Similarly, $f(z;0) \sim z^{\frac{1}{2}+\nu_{UV}}$ for $z \to 0$ and the solution that goes as $z^{\frac{1}{2}-\nu_{UV}}$ must be discarded too. The only way for this solution of (2.38) to exist is that in both cases we remain with the same function. As the "energy" is zero it cannot be quantized as it is usually the case in QM, so $z\,\dot{t}(z)$ can exist only when a fine-tuned relation among parameters in the potential holds, and so also the solution $t(z)$ of (2.6) exists only in this case.

3 The propagator of the boundary theory

Equation (2.6) is invariant under $z \to \lambda z$. A nontrivial domain wall solution spontaneously breaks this invariance, so one expects the appearance of a massless mode, the Goldstone boson in the boundary theory. This was indeed confirmed in [16] for a piece-wise quadratic potential, solving exactly the equation of perturbations (A.6).

On the contrary, applying the standard matching method to find the solution at leading order gives a $1/q^{2\nu_{IR}}$ behavior of the boundary field theory scalar propagator in the $q \to 0$

limit [18]. Puzzled by this discrepancy, we reanalyzed the problem here and we confirm the usual pole behavior for the Goldstone.

We will give now three different derivation of this result, based on exact analytical and approximate techniques. We will find out that in the matching technique the formally next-to-leading order dominates over the formally leading order, thus explaining the discrepancy in the propagator.

Let us now describe the three methods used.

3.1 The piece-wise quadratic potential

This analytic result has been already treated in detail in [16], so we will just shortly summarize the procedure here. The idea is to divide the interesting interval of the scalar field $0 \leq t \leq 1$ into at least three smaller intervals, and write a different quadratic potential in each interval. Special care is taken to guarantee the continuity of the potential and its derivative. In each interval both the equation of motion (through powers of z) and equation for perturbations (through Bessel functions) can be solved analytically. The coefficients in each region and the z-coordinates where the gluing is carried out are determined by the requirement of continuity of the solution and of its derivative. A simple counting can show that due to dilatation invariance there is one equation more than free variables to be determined, here is the technical reason for the fine-tuning. With this method we were able to show analytically the $q \to 0$ limit of the boundary propagator, i.e. the Goldstone pole $1/q^2$.

One could argue that the two results ([16] vs. [18]) are different because they describe different systems. However the differences are actually only apparent.

First, in [16] the potential considered had a minimum at the origin, while the one in [18] has a maximum. It is easy to reproduce the calculations of [16] for the transition from a maximum to a minimum. All the results remain basically unaltered.

Second, although the piece-wise quadratic potential is chosen generic up to the necessary fine-tuning among the parameters explained in the previous section, it can be actually described by a superpotential. We proved this numerically for some specific numerical inputs, and the results are at least apparently indistinguishable from the analytic solution of the full second order equation.

Third, the system in [18] had a dynamical gravity, while our system in [16] considered the $\kappa \to 0$ limit of a fixed AdS background. But, as we learned in section 2.2, this limit is smooth, so that a totally different result for the propagator looks unlikely.

Finally, the technique of [18] can be used also with a general potential that cannot be written through a superpotential, and applying the matching method at the leading order the same $1/q^{2\nu_{IR}}$ propagator comes out. So, it seems likely that one of the two methods used is not completely correct. We will show in the next subsection that the matching technique should be used at next-to-leading order to resolve this particular issue.

3.2 The matching method

To compute the two-function from holography we need to solve equation (A.6)

$$z^2\,\ddot{\xi}(z;q) - (d-1)\,z\,\dot{\xi}(z;q) - \left(q^2\,z^2 + V''(t(z))\right)\,\xi(z;q) = 0 \tag{3.1}$$

The idea is to match the large z known solution of the form

$$\xi(z;q) \xrightarrow{z\to\infty} \xi_\infty(z;q) \equiv \frac{2}{\Gamma(\nu_{IR})}\,\left(\frac{q}{2}\right)^{\nu_{IR}}\,z^{\frac{d}{2}}\,K_{\nu_{IR}}(qz) \tag{3.2}$$

with some analytical solution of the perturbation equation for $q = 0$. Fortunately in the problem considered such solution of (3.1) for $q = 0$ is known. As noted in Subsection 2.4 due to dilatation invariance of the equation of motion (e.o.m.), one solution is

$$\xi_+(z) = z\,\dot{t}(z) \tag{3.3}$$

with $t(z)$ the solution of the e.o.m., while the second can be found from the integral

$$\xi_-(z) = \xi_+(z)\left(\int_{z_i}^{z} dy \frac{y^{d-1}}{\xi_+^2(y)} + \frac{\xi_-(z_i)}{\xi_+(z_i)}\right) \tag{3.4}$$

where z_i and $\xi_-(z_i)$ are integration constants; of course the physics can not depend on the choice of them.

The idea is to match at some large z, but small qz, the two solutions, i.e. determine the ratio $C_+(q)/C_-(q)$ from the behavior of the leading $z^{d-\Delta^{IR}}$ and $z^{\Delta^{IR}}$ terms of (3.2) and the approximate solution

$$\xi(z;q) \approx C_+(q)\,\xi_+(z) + C_-(q)\,\xi_-(z) \tag{3.5}$$

Since from their definitions (3.3) and (3.4) the two solutions go in the UV as

$$\xi_+(z) \xrightarrow{z\to 0} \Delta^{UV}\,z^{\Delta^{UV}} \qquad ; \qquad \xi_-(z) \xrightarrow{z\to 0} \frac{z^{d-\Delta^{UV}}}{(d-2\,\Delta^{UV})\,\Delta^{UV}} \tag{3.6}$$

the boundary propagator can be approximately calculated for $q \to 0$ from

$$G_2(q) = \left(\Delta^{UV}\right)^2\,\left(d-2\,\Delta^{UV}\right)\,\frac{C_+(q)}{C_-(q)} \tag{3.7}$$

That an overlapping region exists it has been proved in [18], where it was also shown that this method leads (at this order) to the $1/q^{2\nu_{IR}}$ behavior of the propagator at $q \to 0$.

The method implicitly assumes that the approximation (3.5) is good enough. We will show now that this is not the case for the issue of the propagator. Let's see what happens at the next order in q^2. The solution gets expanded as power series in q^2:

$$\xi(z;q) = \sum_{n=0}^{\infty} q^{2n}\,\xi^{(n)}(z;q) \tag{3.8}$$

where now (3.5) is just the leading order

$$\xi^{(0)}(z;q) = C_+^{(0)}(q)\ \xi_+(z) + C_-^{(0)}(q)\ \xi_-(z) \tag{3.9}$$

and the $n-$th term solves

$$z^2\ \ddot{\xi}^{(n)}(z;q) - (d-1)\ z\ \dot{\xi}^{(n)}(z;q) - V''(t(z))\ \xi^{(n)}(z;q) = z^2\ \xi^{(n-1)}(z;q) \qquad n = 1, 2, \ldots \tag{3.10}$$

Let us solve it for $n = 1$:

$$\begin{aligned} \xi^{(1)}(z;q) &= C_+^{(1)}(q)\ \xi_+(z) + C_-^{(1)}(q)\ \xi_-(z) \\ &+ C_+^{(0)}(q)\ \xi_+(z) \int_{z_i}^{z} dx\, \frac{x^{d-1}}{\xi_+^2(x)} \int_{z_i}^{x} dy\, \frac{\xi_+^2(y)}{y^{d-1}} \\ &+ C_-^{(0)}(q)\ \xi_-(z) \int_{z_i}^{z} dx\, \frac{x^{d-1}}{\xi_-^2(x)} \int_{z_i}^{x} dy\, \frac{\xi_-^2(y)}{y^{d-1}} \end{aligned} \tag{3.11}$$

where the first two terms on the right-hand-side represent the general solution of the homogeneous equation, while the last two terms are particular solutions of the non-homogeneous equation. In deriving it we took into account the linearity of the equation (3.10).

The next-to-leading order solution is thus

$$\begin{aligned} \xi(z;q) &\approx \xi_+(z)\ \left[\left(C_+^{(0)}(q) + q^2\, C_+^{(1)}(q)\right) + q^2\, C_+^{(0)}(q) \int_{z_i}^{z} dx\, \frac{x^{d-1}}{\xi_+^2(x)} \int_{z_i}^{x} dy\, \frac{\xi_+^2(y)}{y^{d-1}}\right] \\ &+ \xi_-(z)\ \left[\left(C_-^{(0)}(q) + q^2\, C_-^{(1)}(q)\right) + q^2\, C_-^{(0)}(q) \int_{z_i}^{z} dx\, \frac{x^{d-1}}{\xi_-^2(x)} \int_{z_i}^{x} dy\, \frac{\xi_-^2(y)}{y^{d-1}}\right] \end{aligned} \tag{3.12}$$

At this order of the q^2 expansion we can replace

$$q^2\, C_\pm^{(0)} \to q^2\ \left(C_\pm^{(0)}(q) + q^2\, C_\pm^{(1)}(q)\right) \tag{3.13}$$

So we are left with just two integration constants, denoted from now on by

$$C_\pm(q) \equiv C_\pm^{(0)}(q) + q^2\, C_\pm^{(1)}(q) \tag{3.14}$$

as it should be for a second order differential equation. The solution is now

$$\begin{aligned} \xi(z;q) &\approx C_+(q)\ \xi_+(z) \left(1 + q^2 \int_{z_i}^{z} dx\, \frac{x^{d-1}}{\xi_+^2(x)} \int_{z_i}^{x} dy\, \frac{\xi_+^2(y)}{y^{d-1}}\right) \\ &+ C_-(q)\ \xi_-(z) \left(1 + q^2 \int_{z_i}^{z} dx\, \frac{x^{d-1}}{\xi_-^2(x)} \int_{z_i}^{x} dy\, \frac{\xi_-^2(y)}{y^{d-1}}\right) \end{aligned} \tag{3.15}$$

Finally, for later use, the double integral can be simplified as

$$\xi_\pm(z) \int_{z_i}^{z} dx \frac{x^{d-1}}{\xi_\pm^2(x)} \int_{z_i}^{x} dy \frac{\xi_\pm^2(y)}{y^{d-1}} = \pm \xi_\mp(z) \int_{z_i}^{z} dx \frac{\xi_\pm^2(x)}{x^{d-1}} \mp \xi_\pm(z) \int_{z_i}^{z} dx \frac{\xi_+(x)\xi_-(x)}{x^{d-1}} \tag{3.16}$$

where we used the relation

$$\frac{z^{d-1}}{\xi_\pm^2(z)} = \pm \left(\frac{\xi_\mp(z)}{\xi_\pm(z)}\right)' \tag{3.17}$$

that follows from the definitions (3.3), (3.4).

	a_+	a_-
UV	Δ	$\frac{1}{\Delta(d-2\Delta)}$
IR	Δ	$\frac{1}{\Delta(d+2\Delta)}$

Table 1: The explicit values in (3.21) and (3.22).

3.2.1 An explicit calculation

We will consider here the case of subsection 2.2.1 with the solution to the equation of motion

$$t(z) = \frac{z^\Delta}{1+z^\Delta} \tag{3.18}$$

The corresponding solutions to the equation of perturbation at $q=0$ are

$$\xi_+(z) = \frac{\Delta z^\Delta}{(1+z^\Delta)^2} \tag{3.19}$$

$$\xi_-(z) = \frac{1}{\Delta(1+z^\Delta)^2}\sum_{k=0}^{4}\binom{4}{k}\frac{z^{d+(k-1)\Delta}}{d+(k-2)\Delta} \tag{3.20}$$

with the limits,

$$\xi_+(z) \xrightarrow{z\to 0} a_+^{UV}\ z^\Delta \qquad ; \qquad \xi_-(z) \xrightarrow{z\to 0} a_-^{UV}\ z^{d-\Delta} \tag{3.21}$$

$$\xi_+(z) \xrightarrow{z\to \infty} a_+^{IR}\ z^{-\Delta} \qquad ; \qquad \xi_-(z) \xrightarrow{z\to \infty} a_-^{IR}\ z^{d+\Delta} \tag{3.22}$$

where the explicit values of the constants are given in Table 1.

We can evaluate the integrals as

$$\int_{z_i}^{z} dx \frac{\xi_+^2(x)}{x^{d-1}} = \Delta\ B_{(t,t_i)}\left(2-\frac{d-2}{\Delta}, 2+\frac{d-2}{\Delta}\right) \tag{3.23}$$

$$\int_{z_i}^{z} dx \frac{\xi_+(z)\xi_-(x)}{x^{d-1}} = \frac{1}{\Delta}\sum_{k=0}^{4}\frac{1}{d+(k-2)\Delta}\binom{4}{k} B_{(t,t_i)}\left(\frac{2}{\Delta}+k, 4-\frac{2}{\Delta}-k\right) \tag{3.24}$$

$$\begin{aligned}\int_{z_i}^{z} dx \frac{\xi_-^2(x)}{x^{d-1}} &= \frac{1}{\Delta^3}\sum_{k,l=0}^{4}\frac{1}{d+(k-2)\Delta}\binom{4}{k}\frac{1}{d+(l-2)\Delta}\binom{4}{l} \\ &\times\ B_{(t,t_i)}\left(\frac{d+2}{\Delta}+k+l-2, -\frac{d+2}{\Delta}-k-l+6\right)\end{aligned} \tag{3.25}$$

where $t=t(z)$, $t_i = t(z_i)$ and

$$B_{(t,t_i)}(a,b) \equiv \int_{t_i}^{t} d\tau \tau^{a-1}(1-\tau)^{b-1} \tag{3.26}$$

is the generalized incomplete (Euler) beta function. They can be expanded as

$$\text{IR } (t \to 1) \quad : \quad B_{(t,t_i)}(a,b) = -\sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(a) \left((1-t)^{b+n} - (1-t_i)^{b+n}\right)}{(b+n)\Gamma(n+1)\Gamma(a-n)} \tag{3.27}$$

$$\text{UV } (t \to 0) \quad : \quad B_{(t,t_i)}(a,b) = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(b) \left(t^{a+n} - t_i^{a+n}\right)}{(a+n)\Gamma(n+1)\Gamma(b-n)} \tag{3.28}$$

Now we can combine the different expansions to get the needed powers:

$$\xi_+(z) \quad \to \quad a_+^{IR} z^{-\Delta} \tag{3.29}$$

$$\xi_-(z) \quad \to \quad a_-^{IR} z^{d+\Delta} \tag{3.30}$$

$$\xi_+(z) \int_{z_i}^{z} dx \frac{x^{d-1}}{\xi_+^2(x)} \int_{z_i}^{x} dy \frac{\xi_+^2(y)}{y^{d-1}} \quad \to \quad -\epsilon_{+-}^{IR} a_+^{IR} z^{-\Delta} + \epsilon_{++}^{IR} a_-^{IR} z^{d+\Delta} \tag{3.31}$$

$$\xi_-(z) \int_{z_i}^{z} dx \frac{x^{d-1}}{\xi_-^2(x)} \int_{z_i}^{x} dy \frac{\xi_-^2(y)}{y^{d-1}} \quad \to \quad -\epsilon_{--}^{IR} a_+^{IR} z^{-\Delta} + \epsilon_{+-}^{IR} a_-^{IR} z^{d+\Delta} \tag{3.32}$$

with

$$\epsilon_{++}^{IR} = \Delta B_{(1,t_i)}(2-(d-2)/\Delta, 2+(d-2)/\Delta) \tag{3.33}$$

$$\epsilon_{+-}^{IR} = \sum_{k=0}^{4} \binom{4}{k} \frac{B_{(1,t_i)}(2/\Delta+k, 4-2/\Delta-k)}{\Delta(d+(k-2)\Delta)} \tag{3.34}$$

$$\begin{aligned} \epsilon_{--}^{IR} &= \sum_{k,l=0}^{4} \frac{1}{d+(k-2)\Delta} \binom{4}{k} \frac{1}{d+(l-2)\Delta} \binom{4}{l} \\ &\times \frac{B_{(1,t_i)}(-2+(d+2)/\Delta+k+l, 6-(d+2)/\Delta-k-l)}{\Delta^3} \end{aligned} \tag{3.35}$$

where we used

$$B_{(t_2,t_1)}(a,b) = B_{(1-t_1,1-t_2)}(b,a) \tag{3.36}$$

We get close to $z = \infty$

$$\begin{aligned} \xi(z;q) \quad \to \quad & \left[\left(1 - q^2 \epsilon_{+-}^{IR}\right) C_+(q) - q^2 \epsilon_{--}^{IR} C_-(q)\right] a_+^{IR} z^{-\Delta} \\ + \quad & \left[\left(1 + q^2 \epsilon_{+-}^{IR}\right) C_-(q) + q^2 \epsilon_{++}^{IR} C_+(q)\right] a_-^{IR} z^{d+\Delta} \end{aligned} \tag{3.37}$$

The system to solve after matching to (3.2) is

$$a_+^{IR} \left[\left(1 - q^2 \epsilon_{+-}^{IR}\right) C_+(q) + q^2 \epsilon_{--}^{IR} C_-(q)\right] = 1 \tag{3.38}$$

$$a_-^{IR} \left[\left(1 + q^2 \epsilon_{+-}^{IR}\right) C_-(q) + q^2 \epsilon_{++}^{IR} C_+(q)\right] = \gamma q^{2\nu_{IR}} \tag{3.39}$$

where

$$\gamma \equiv \frac{\Gamma(-\nu_{IR})}{2^{2\nu_{IR}} \Gamma(\nu_{IR})} \tag{3.40}$$

and where in this case

$$\nu_{IR} = \Delta + d/2 \tag{3.41}$$

The leading order solution for small q is

$$C_+(q) = \frac{1}{a_+^{IR}} + \dots \tag{3.42}$$

$$C_-(q) = -\epsilon_{++}^{IR} q^2 C_+(q) + \dots = -\frac{\epsilon_{++}^{IR}}{a_+^{IR}} q^2 + \dots \tag{3.43}$$

If we kept only the formally leading order expression, i.e. no ϵ's in (3.38)-(3.39), we would get a wrong result: although $C_+(q)$ remains the same, $C_-(q) = (\gamma/a_-^{IR})q^{2\nu_{IR}}$ changes drastically.

For $z \to 0$ on the other side

$$\xi_+(z) \to a_+^{UV} z^{\Delta} \tag{3.44}$$

$$\xi_-(z) \to a_-^{UV} z^{d-\Delta} \tag{3.45}$$

$$\xi_+(z)\int_{z_i}^{z} dx \frac{x^{d-1}}{\xi_+^2(x)} \int_{z_i}^{x} dy \frac{\xi_+^2(y)}{y^{d-1}} \to -\epsilon_{+-}^{UV} a_+^{UV} z^{\Delta} + \epsilon_{++}^{UV} a_-^{UV} z^{d-\Delta} \tag{3.46}$$

$$\xi_-(z)\int_{z_i}^{z} dx \frac{x^{d-1}}{\xi_-^2(x)} \int_{z_i}^{x} dy \frac{\xi_-^2(y)}{y^{d-1}} \to -\epsilon_{--}^{UV} a_+^{UV} z^{\Delta} + \epsilon_{+-}^{UV} a_-^{UV} z^{d-\Delta} \tag{3.47}$$

with

$$\epsilon_{++}^{UV} = -\Delta B_{(t_i,0)}(2-(d-2)/\Delta, 2+(d-2)/\Delta) \tag{3.48}$$

$$\epsilon_{+-}^{UV} = -\sum_{k=0}^{4}\binom{4}{k}\frac{B_{(t_i,0)}(2/\Delta + k, 4 - 2/\Delta - k)}{\Delta(d+(k-2)\Delta)} \tag{3.49}$$

$$\begin{aligned}\epsilon_{--}^{UV} &= -\sum_{k,l=0}^{4}\frac{1}{d+(k-2)\Delta}\binom{4}{k}\frac{1}{d+(l-2)\Delta}\binom{4}{l} \\ &\times \frac{B_{(t_i,0)}(-2+(d+2)/\Delta + k + l, 6-(d+2)/\Delta - k - l)}{\Delta^3}\end{aligned} \tag{3.50}$$

so that

$$\begin{aligned}\xi(z;q) \to\ & \left[\left(1 - q^2\epsilon_{+-}^{UV}\right)C_+(q) - q^2\epsilon_{--}^{UV}C_-(q)\right] a_+^{UV} z^{\Delta} \\ +\ & \left[\left(1 + q^2\epsilon_{+-}^{UV}\right)C_-(q) + q^2\epsilon_{++}^{UV}C_+(q)\right] a_-^{UV} z^{d-\Delta}\end{aligned} \tag{3.51}$$

The propagator is then,

$$G_2(q) \approx \frac{a_+^{UV}\left[\left(1 - q^2\epsilon_{+-}^{UV}\right)C_+(q) - q^2\epsilon_{--}^{UV}C_-(q)\right]}{a_-^{UV}\left[\left(1 + q^2\epsilon_{+-}^{UV}\right)C_-(q) + q^2\epsilon_{++}^{UV}C_+(q)\right]} \to \frac{a_+^{UV}/a_-^{UV}}{\epsilon_{++}^{UV} - \epsilon_{++}^{IR}} \times \frac{1}{q^2} \tag{3.52}$$

In our concrete example this is

$$G_2(q) = \frac{\alpha}{q^2} \tag{3.53}$$

with

$$\alpha = \frac{\Delta(2\Delta - d)}{B_{(1,0)}(2-(d-2)/\Delta, 2+(d-2)/\Delta)} \tag{3.54}$$

which is strictly positive for any $\Delta > d/2$. This confirms the Goldstone theorem.

The final result is independent on the arbitrary parameter $t_i = t(z_i)$, as it should be, although several intermediate quantities depend on it. Needless to say, the result exactly coincides with the general formulae from appendix A.5.

On the other side, if we had retained only the formally leading order in q^2, we would get a wrong limit for the propagator,

$$G_2(q) \approx \left[(a_+^{UV}/a_-^{UV})(a_-^{IR}/a_+^{IR})(1/\gamma)\right] q^{-2\nu_{IR}} \tag{3.55}$$

3.2.2 Beyond $q \to 0$

What we derived is strictly speaking valid exactly only for $q \to 0$. For all z we can then use the solution for infinitely small q, i.e. (3.15). What if q is small but non-zero? Can we use the same solution (3.15)?

The point is that for a finite q even matching must be done at a finite z. In fact one must find a region in the $z-q$ plane where both approximations are valid. More precisely, for large enough $z > z_\infty$ (see below) the small q approximate solution (3.15) (or a better one) is valid for $z \lesssim z_0(q)$ where[8]

$$z_0(q) \equiv |V''(1)|^{1/2}/q \tag{3.56}$$

while the large z solution (3.2) (or a better one) is valid for $z \gtrsim z_\infty$ (see Appendix A.1):

$$z_\infty \equiv \left|\frac{V''(1)}{V'''(1)\; a_{IR}}\right|^{\frac{1}{\Delta^{IR}}} \tag{3.57}$$

Then one has to match the two solutions not at $z \to \infty$ as we did in the previous example but at a finite although large enough $z_\infty \lesssim z \lesssim z_0(q)$, which may modify the values of $C_\pm(q)$. Clearly, in order for such a region to exist at all, $z_\infty < z_0(q)$, which gives an upper bound for q from the solution of $z_\infty = z_0(q_{max})$. For higher q the method fails.

So the matching depends on q, although once we match for a given q, let us call it $\bar{q}$, then it is valid for all $q \leq \bar{q}$. Then we can use the solution (3.15) (or a better one) for all $z \lesssim z_0(\bar{q})$ and the solution (3.2) (or a better one) for all $z \gtrsim z_\infty$. The situation is summarized on fig. 1.

3.3 The WKB approximation method

Here we shall try to apply the WKB method in order to compute the two-point correlation function. The straightest way of doing it is to consider the Schrödinger-type equation

[8] except for small regions in which $V''(t(z)) \approx 0$.

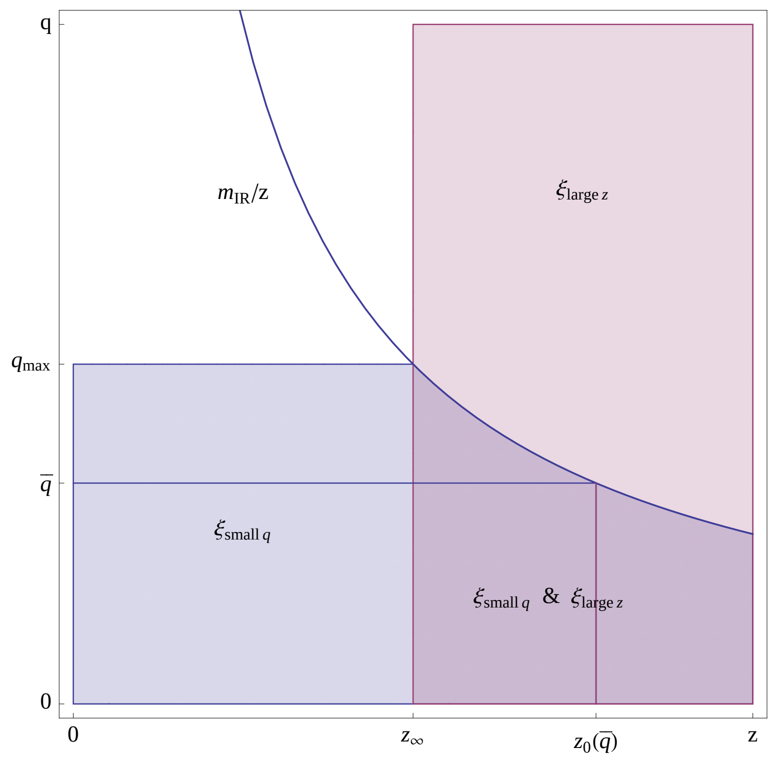

Figure 1: A schematic diagram of the z-q plane divided into regions in which different approximations are valid. $\xi_{large\ z}$ is approximated by (3.2), $\xi_{small\ q}$ by (3.15), and $m_{IR} \equiv |V''(1)|^{1/2}$. For any $\bar{q} \leq q_{max}$ the same approximate solutions can be used. In the white region all approximations mentioned in this paper fail.

(2.38) with "potential"

$$Q(z;q) \equiv q^2 + \frac{1}{z^2}\left(\frac{d^2-1}{4} + V''(t(z))\right) \tag{3.58}$$

where we remember that $t(z)$ is the solution of (2.6). For simplicity we consider the case $V''(0) \equiv m_{UV}^2 > 0$, although it is not necessary for the argument.

The WKB approximation results a good one if the slowly varying "Compton length" condition holds,

$$\left|\frac{d|Q(z;q)|^{-\frac{1}{2}}}{dz}\right| = \left|\frac{\dot{Q}(z;q)}{2\,|Q(z;q)|^{\frac{3}{2}}}\right| \ll 1 \tag{3.59}$$

This condition applied to (3.58) reads,

$$\frac{\left|\frac{d^2-1}{4} + V''(t(z)) - \frac{1}{2}\,V'''(t(z))\,z\,\dot{t}(z)\right|}{\left|\frac{d^2-1}{4} + V''(t(z)) + q^2\,z^2\right|^{\frac{3}{2}}} \ll 1 \tag{3.60}$$

From here is straightforward to see that the WKB solution is trustable for any q^2 around $z = 0$ and $z = \infty$ if,

$$\nu_{UV} \gg \frac{1}{2} \qquad ; \qquad \nu_{IR} \gg \frac{1}{2} \tag{3.61}$$

respectively, with $\nu_{UV/IR}$ as in (2.8). Furthermore, $Q(z;q)$ is positive near $z = 0$ (and diverges quadratically there), but it is also positive for large z (going to q^2 from above). What happens in the middle? From subsection 2.3 we know that for q small enough $Q(z;q)$ must become negative; then for some z_M where $t(z_M) = t_M$ it should have a local minimum. Then there must exist $z_i = z_i(q)$, $z_1(q) < z_M < z_2(q)$ such that,

$$z_i{}^2\, Q(z_i;q) = \frac{d^2-1}{4} + V''(t(z_i)) + q^2\, z_i{}^2 = 0 \quad ; \quad i = 1,2 \tag{3.62}$$

Near these zeroes of $Q(z;q)$ the WKB approximation breaks down.

If we admit that $V''(t_M)$ is large enough then it is seen from (3.60) that in the region near z_M the WKB solution is trustable too. Therefore, calling I, II, III the regions near $z = 0$, z_M and $z \gg 1$ respectively, we can write the approximate WKB solution in each region as,

$$\xi_I(z;q) \;=\; C_I^+\, z^{\frac{d}{2}}\, \frac{\exp\left(\int_{z_1}^z \frac{dz}{z}\sqrt{z^2\,Q(z;q)}\right)}{(z^2\,Q(z;q))^{\frac14}} + C_I^-\, z^{\frac{d}{2}}\, \frac{\exp\left(-\int_{z_1}^z \frac{dz}{z}\sqrt{z^2\,Q(z;q)}\right)}{(z^2\,Q(z;q))^{\frac14}} \tag{3.63}$$

$$\xi_{II}(z;q) \;=\; C_{II}\, z^{\frac{d}{2}}\, \frac{\exp\left(i\int_{z_1}^z \frac{dz}{z}\sqrt{-z^2 Q(z;q)}\right)}{(-z^2\,Q(z;q))^{\frac14}} + C_{II}^*\, z^{\frac{d}{2}}\, \frac{\exp\left(-i\int_{z_1}^z \frac{dz}{z}\sqrt{-z^2 Q(z;q)}\right)}{(-z^2\,Q(z;q))^{\frac14}} \tag{3.64}$$

$$\xi_{III}(z;q) \;=\; C_{III}^+\, z^{\frac{d}{2}}\, \frac{\exp\left(\int_{z_2}^z \frac{dz}{z}\sqrt{z^2\,Q(z;q)}\right)}{(z^2\,Q(z;q))^{\frac14}} + C_{III}^-\, z^{\frac{d}{2}}\, \frac{\exp\left(-\int_{z_2}^z \frac{dz}{z}\sqrt{(z^2\,Q(z;q)}\right)}{(z^2\,Q(z;q))^{\frac14}} \tag{3.65}$$

where the coefficients are related by,

$$C_I^\pm \;=\; \frac{1\pm 3}{2}\, Im\left(C_{II}\, e^{\pm i\frac{\pi}{4}}\right) \quad \leftrightarrow \quad C_{II} = \frac{1}{2}\, e^{+i\frac{\pi}{4}}\, C_I^+ + e^{-i\frac{\pi}{4}}\, C_I^- = (C_{II}^*)^* \tag{3.66}$$

$$\begin{aligned} C_{III}^\pm \;&=\; \frac{3\pm 1}{2}\, Im\left(C_{II}\, e^{i(\varphi(q)\mp\frac{\pi}{4})}\right) \quad \leftrightarrow \\ C_{II} \;&=\; e^{-i\varphi(q)}\left(-\frac{1}{2}\, e^{-i\frac{\pi}{4}}\, C_{III}^+ + e^{+i\frac{\pi}{4}}\, C_{III}^-\right) = (C_{II}^*)^* \end{aligned} \tag{3.67}$$

and,

$$\varphi(q) \equiv \int_{z_1(q)}^{z_2(q)} \frac{dz}{z}\, \sqrt{-z^2\,Q(z;q)} \tag{3.68}$$

Now, imposing finiteness when $z \to \infty$ implies $C^+_{III} = 0$. By using the relations (3.66) and (3.67) we get all the constants in terms of C^-_{III}; in particular for the solution near $z = 0$ we get,

$$\xi_I(z;q) = C^-_{III} \left(2\cos\varphi(q)\, z^{\frac{d}{2}} \frac{\exp\left(\int_{z_1(q)}^{z} \frac{dz}{z} \sqrt{z^2\,Q(z;q)}\right)}{(z^2\,Q(z;q))^{\frac{1}{4}}} \right.$$
$$\left. + \quad \sin\varphi(q)\, z^{\frac{d}{2}} \frac{\exp\left(-\int_{z_1(q)}^{z} \frac{dz}{z} \sqrt{z^2\,Q(z;q)}\right)}{(z^2\,Q(z;q))^{\frac{1}{4}}} \right) \tag{3.69}$$

From here we should be able to extract the propagator as a function of q^2, at least for q not so large. But as we know, for $q = 0$ (3.69) must be equal to $z\,t'(z)$ and thus going only as $z^{\Delta^{UV}}$ for $z \to 0$. We will show now that this implies the constraint $\varphi(0) = k\,\pi$ with k an integer. First we rewrite

$$\exp\left(\pm\int_{z_1(q)}^{z} \frac{dz}{z}\sqrt{z^2\,Q(z;q)}\right) = \left(\frac{z}{z_1(q)}\right)^{\pm\sqrt{\nu_{UV}^2-1/4}} \tag{3.70}$$
$$\times \quad \exp\left(\pm\int_{z_1(q)}^{z} \frac{dz}{z}\left(\sqrt{z^2\,Q(z;q)} - \sqrt{\nu_{UV}^2 - 1/4}\right)\right)$$

Since we are interested only in $\nu_{UV} \gg 1/2$ and leading behavior at $z \to 0$, we can see with the help of (3.70) that the first term on the r.h.s. of (3.69) goes like $z^{\Delta^{UV}}$, while the second goes like $z^{d-\Delta^{UV}}$. Since this last one should not be present in the solution $z\,t'(z)$ of the $q = 0$ perturbation, we have to impose (otherwise no solution with the right asymptotic behavior exists)

$$\varphi(0) \equiv \int_{z_1(0)}^{z_2(0)} \frac{dz}{z}\sqrt{-z^2\,Q(z;0)} = k\,\pi \tag{3.71}$$

This means that only potentials which satisfy this constraint are acceptable. This is the WKB analog of the fine-tuning mentioned before.

This simple conclusion is the reason for the $1/q^2$ behavior of the boundary propagator. In fact, it is easy to derive the form of the propagator in the WKB approximation:

$$G_2(q) = \frac{2\exp\left(-2\int_0^{z_1(q)} \frac{dz}{z}\left(\sqrt{z^2\,Q(z;q)} - \sqrt{\nu_{UV}^2 - 1/4}\right)\right)}{(z_1(q))^{2\sqrt{\nu_{UV}^2-1/4}}\tan\varphi(q)} \tag{3.72}$$

Clearly, due to (3.71), we get for $q \to 0$ the usual Goldstone pole

$$G_2(q) \approx \frac{2\exp\left(-2\int_0^{z_1(0)} \frac{dz}{z}\left(\sqrt{z^2 Q(z;0)} - \sqrt{\nu_{UV}^2 - 1/4}\right)\right)}{(z_1(0))^{2\sqrt{\nu_{UV}^2-1/4}}(d\varphi(q)/dq^2)_{q^2=0}} \times \frac{1}{q^2} \tag{3.73}$$

where

$$\left.\frac{d\varphi(q)}{dq^2}\right|_{q^2=0} = -\frac{1}{2}\int_{z_1(0)}^{z_2(0)} dz \frac{z}{\sqrt{-z^2 Q(z;0)}} \tag{3.74}$$

Although the denominator vanishes at the integration boundaries, the integral itself is finite.

4 Conclusions and outlook

We considered in this work the bulk system of a real scalar field in a non-dynamical AdS background. After finding BPS-type solutions of the bulk equation of motion, we solved the perturbation equation on this background in two different limits, the large z and the small q regimes. We have shown that a correct application of the matching procedure between these two approximate solutions leads to the $1/q^2$ boundary propagator, as expected by the Goldstone theorem applied to the spontaneous breaking of dilatation invariance.

Is the result of the simple pole propagator in the boundary theory a consequence of the no-backreaction ($\kappa \to 0$) limit? We believe that this is not the case, and that the result is generic. In fact, both at finite or vanishing κ the formally leading order of the matching method gives the $1/q^{2\nu_{IR}}$ behavior, which indicates that the problem is in the application of the method and not in the difference of the systems.

As we saw, there is a maximal momentum for which we can apply the matching method given a potential. The result is in the form of a positive power expansion in q. However, the limitation is due to the vanishing of a common region for the low q and large z expansions. It is thus possible that an analytic continuation of the result exists for all q, although, due to the perturbative character of the solution, it is hard to find it.

On top of all this we would like to stress two interesting results obtained and proved for the bulk potential. First, any potential defined through a superpotential must necessarily have a negative enough second derivative in at least some region between the two extrema. It is not clear to us whether this persists also for more general potentials. Second, we proved in general that a potential must have some correlation (fine-tuning) among parameters in order for the solution to exist.

Before concluding let us add two remarks.

In the approximations used in this paper we can calculate the on-shell 1-particle irreducible 3-point correlator (dilaton)^3. We find it vanishing, as required for a Goldstone.

Finally, there have been several discussions on the role of the a-theorem [21, 22, 23] in holography (see for example [24], [18] and references therein). The value (especially the positivity) of the $\mathcal{O}(q^4)$ coefficient [21] in the $2 \to 2$ dilaton scattering amplitude represents a check of the AdS/CFT correspondence in connection with the a-theorem. Unfortunately, the calculation explicitly involves the bulk-bulk propagator, which we miss at the moment. We leave this interesting issue for the future.

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A The matching method to all orders

Let $t(z)$ be the solution of the equation of motion (2.6) that behaves for $z \to 0$ (UV) and $z \to \infty$ (IR) as,

$$t(z) \xrightarrow{z\to 0} a_{UV}\, z^{\Delta_+^{UV}}\,(1+b_{UV}\, z^{\alpha_{UV}}+\ldots) \qquad ; \qquad t(z) \xrightarrow{z\to\infty} 1+a_{IR}\, z^{\Delta_-^{IR}}\,(1+b_{IR}\, z^{\alpha_{IR}}+\ldots) \tag{A.1}$$

respectively. Here $\alpha_{UV} > 0$ and $\alpha_{IR} < 0$, while that

$$\Delta_\pm^{UV/IR} \equiv \frac{d}{2} \pm \nu_{UV/IR} \qquad ; \qquad \nu_{UV/IR} \equiv \sqrt{\frac{d^2}{4} + m^2_{UV/IR}} \tag{A.2}$$

with $m^2_{UV} \equiv V''(0)$ and $m^2_{IR} \equiv V''(1) > 0$. Note that in order for $t(z)$ to be finite in the asymptotic expansions (A.1) neither Δ_-^{UV} appears in the UV nor Δ_+^{IR} in the IR.

Let us introduce for further use the following expansions of the functions $\xi_\pm(z)$

$$\xi_\pm(z) = a_\pm^{UV/IR}\, z^{\Delta_{\pm/\mp}^{UV/IR}}\, \bar{\xi}_\pm^{UV/IR}(z) \qquad ; \qquad \bar{\xi}_\pm^{UV/IR}(z) \xrightarrow{z\to 0/\infty} 1 \tag{A.3}$$

that follow by plugging (A.1) in the definitions (3.3) and (3.4), where

$$a_+^{UV/IR} \equiv a_{UV/IR}\, \Delta_{+/-}^{UV/IR} \qquad ; \qquad a_-^{UV/IR} \equiv \left(a_+^{UV/IR}\,(d - 2\,\Delta_{+/-}^{UV/IR})\right)^{-1} \tag{A.4}$$

Clearly the UV/IR expansion of $\xi_+(z)$ can not contain the $z^{\Delta_{-/+}^{UV/IR}}$-power, but $\xi_-(z)$ could contain the $z^{\Delta_{+/-}^{UV/IR}}$-power.

Our aim is to solve the equation for perturbations around the solution $t(z)$, i.e. if we write (for a general treatment see for example the appendix of [16])

$$t(z;q) \equiv t(z) + \xi(z;q)\, e^{iq\cdot\frac{x}{L}} \tag{A.5}$$

then to first order in $\xi(z;q)$ (2.3) gives

$$z^2\, \ddot{\xi}(z;q) - (d-1)\, z\, \dot{\xi}(z;q) - \left(q^2\, z^2 + V''(t(z))\right)\, \xi(z;q) = 0 \tag{A.6}$$

We will do it in two different approximations.

A.1 The large z expansion.

We write (A.6) as

$$z^2\, \ddot{\xi}(z;q) - (d-1)\, z\, \dot{\xi}(z;q) - \left(q^2\, z^2 + m_{IR}^2\right)\, \xi(z;q) = \delta(z)\, \xi(z;q) \tag{A.7}$$

and consider $\delta(z) \equiv V''(t(z)) - V''(1)$ small in the sense,

$$|\delta(z)| = |V''(t(z)) - V''(1))| \ll V''(1) \quad \longrightarrow \quad z > z_\infty \equiv \left|\frac{V''(1)}{V'''(1)\, a_{IR}}\right|^{\frac{1}{\Delta_-^{IR}}} \tag{A.8}$$

independently of the value of q. Then the solution for $z > z_\infty$ can be hopefully expanded in orders of $\delta(z)$,

$$\delta(z) = V'''(1)\, a_{IR}\, z^{\Delta_-^{IR}}\, \left(1 + b_{IR}\, z^{\alpha_{IR}} + \dots\right) \tag{A.9}$$

The order zero term is the solution to the l.h.s. of (A.7) equal to zero, which is given by,

$$\xi_\infty(z;q) \equiv \frac{2}{\Gamma(\nu_{IR})}\, \left(\frac{q}{2}\right)^{\nu_{IR}}\, z^{\frac{d}{2}}\, K_{\nu_{IR}}(qz) \tag{A.10}$$

where we have dropped the solution that diverges in the IR and fixed the normalization in such a way that $\xi_\infty(z;0) = z^{\Delta_-^{IR}}$. It is not difficult to see that the expansion for large $z > z_\infty$ is of the form,

$$\xi(z;q) = \xi_\infty(z;q)\, \left(1 + \frac{f_0(qz)}{z^{-\Delta_-^{IR}}} + \dots\right) \tag{A.11}$$

where for completeness we quote the first correction,

$$f_0(u) = V'''(1)\, a_{IR}\, u^{-\Delta_-^{IR}}\, \int_\infty^u \frac{dx}{x\, K_{\nu_{IR}}^2(x)} \int_\infty^x \frac{dy}{y^{1-\Delta_-^{IR}}}\, K_{\nu_{IR}}^2(y) \tag{A.12}$$

However corrections to the leading term of $\xi(z;q)$ in negative powers of z will not be relevant in the matching procedure, at least not to compute the leading order behavior of the two-point function.

A.2 The small q expansion.

This time we write (A.6) as

$$z^{d-1}\, \frac{d}{dz}\left(z^{1-d} \frac{d\xi(z;q)}{dz}\right) - \frac{V''(t(z))}{z^2}\, \xi(z;q) = q^2\, \xi(z;q) \tag{A.13}$$

and consider q small in the sense,

$$q \ll \frac{|V''(t(z))|^{\frac{1}{2}}}{z} \tag{A.14}$$

This condition certainly holds in the UV region near $z = 0$, but also in the IR region if

$$q^2\, z^2 \ll |V''(t(z)| \sim m_{IR}^2 \quad \longrightarrow \quad qz \ll m_{IR} \tag{A.15}$$

that is, when z is large and q small but qz fixed and small enough.

Under this condition we can try a solution for small q as a power series in q^2,

$$\xi(z;q) = \sum_{m\geq 0} q^{2m}\, \xi^{(m)}(z;q) \tag{A.16}$$

Plugging this expansion in (A.13) we get,

$$z^{d-1}\,\frac{d}{dz}\left(z^{1-d}\,\frac{d\xi^{(0)}(z;q)}{dz}\right) - \frac{V''(t(z))}{z^2}\,\xi^{(0)}(z;q) \;=\; 0 \tag{A.17}$$

$$z^{d-1}\,\frac{d}{dz}\left(z^{1-d}\,\frac{d\xi^{(m)}(z;q)}{dz}\right) - \frac{V''(t(z))}{z^2}\,\xi^{(m)}(z;q) \;=\; \xi^{(m-1)}(z;q) \quad ; \quad m = 1,2,\ldots \tag{A.18}$$

The solution to lowest order is,

$$\xi^{(0)}(z;q) = C_+^{(0)}(q)\;\xi_+(z) + C_-^{(0)}(q)\;\xi_-(z) \tag{A.19}$$

where $C_\pm^{(0)}(q)$ are integration constants. From $\xi^{(0)}(z;q)$ we can determine $\xi^{(m)}(z;q)$ from (A.18), and so on.

This iterative procedure yields the solution in the following form. First we introduce the set of functions,

$$\begin{aligned} f_{ij}^{(k)}(z) \;&\equiv\; \int_{z_i}^{z} \frac{dw}{w^{d-1}}\,\xi_i(w)\;\xi_j^{(k)}(w) \quad ; \quad i,j = +,- \quad , \quad k = 0,1,\ldots \\ \xi_\pm^{(0)}(z) \;&\equiv\; \xi_\pm(z) \end{aligned} \tag{A.20}$$

where

$$\xi_\pm^{(k)}(z) \equiv -f_{-\pm}^{(k-1)}(z)\;\xi_+(z) + f_{+\pm}^{(k-1)}(z)\;\xi_-(z) \quad , \quad k = 1,2,\ldots \tag{A.21}$$

All of them are obtained iteratively: first, from (A.20) with $k = 0$ we get $f_{ij}^{(0)}(z)$, then we go to (A.21) with $k = 1$ and get $\xi_\pm^{(1)}(z)$, then we come back to (A.20) with $k = 1$ and get $f_{ij}^{(1)}(z)$ and so on. The functions $\xi_m(z;q)$ can be expressed in terms of the $\xi_\pm^{(k)}(z)$'s yielding the full expansion (A.16) in the form,

$$\xi(z;q) = \sum_{m\geq 0} q^{2m} \sum_{k=0}^{m} \left(C_+^{(m-k)}(q)\;\xi_+^{(k)}(z) + C_-^{(m-k)}(q)\;\xi_-^{(k)}(z)\right) \tag{A.22}$$

where the $C_\pm^{(k)}$'s are, as in (A.19), the integration constants of the homogeneous solution in (A.18). After some rearrangement, we can write (A.22) as,

$$\xi(z;q) = C_+(q) \sum_{m\geq 0} q^{2m}\;\xi_+^{(m)}(z) + C_-(q) \sum_{m\geq 0} q^{2m}\;\xi_-^{(m)}(z) \tag{A.23}$$

where we have redefined the coefficients

$$C_{\pm}(q) \equiv \sum_{k\geq 0} C_{\pm}^{(k)}(q)\, q^{2k} \tag{A.24}$$

We should not be surprised of this expression; after all (A.13) is a second order linear differential equation and both sums in (A.23) are linearly independent solutions of it as it can be quickly checked. Note furthermore that they are holomorphic in q^2; the reason behind this fact can be traced directly to the assumption (A.16).

A.3 The two-point function.

For z small, more explicitly $qz \ll m_{UV}$ expansion (A.23) hopefully holds, and it can be used to compute the two-point correlation function as follows. After adjusting the constant of integration in (A.1) to get rid of the $z^{\Delta_+^{UV}}$ term in $\xi_-(z)$, we parametrize the $z \to 0$ behavior as

$$\begin{aligned} \xi(z;q) \;\; &\to \;\; \left[\left(1 - q^2\epsilon_{-+}^{UV}(q)\right) C_+(q) - q^2\epsilon_{--}^{UV}(q) C_-(q)\right] a_+^{UV} z^{\Delta_+^{UV}} + \ldots \\ &+ \;\; \left[\left(1 + q^2\epsilon_{+-}^{UV}(q)\right) C_-(q) + q^2\epsilon_{++}^{UV}(q) C_+(q)\right] a_-^{UV} z^{\Delta_-^{UV}} + \ldots \end{aligned} \tag{A.25}$$

where

$$\epsilon_{+\pm}^{UV}(q) = -\sum_{m\geq 0} q^{2m} \int_0^{z_i} dw\; w^{1-d}\; \xi_+(w)\; \xi_{\pm}^{(m)}(w) \tag{A.26}$$

while we were unable to find a closed expression for $\epsilon_{-\pm}^{UV}$ without specifying the potential.

Applying the holographic recipe the two-point function results,

$$G_2(q) \xrightarrow{q\to 0} \frac{a_+^{UV}/a_-^{UV}}{\left(q^2\; \epsilon_{++}^{UV}(q) + \frac{C_-(q)}{C_+(q)}\right)_{q\to 0}} \tag{A.27}$$

The knowledge of the leading order behavior of the quotient $C_-(q)/C_+(q)$ for $q \to 0$ will allow to compute the leading power in q of $G_2(q)$. The z_i-dependence of the coefficients $\epsilon_{++}^{UV}(q)$ (and the z_i-independence of the physics) gives a hint that this power is -2, as we will confirm below.

A.4 The infrared expansion

Here we define the functions $\bar{F}_{ij}^{(m)}(z)$ and the constants $\bar{\varphi}_{ij}^{(m)}$ by means of the integrals,

$$\begin{aligned} &a_i^{IR}\, a_j^{IR}\, \sigma_j^{(m)} \int_{z_i}^{z} dw\; w^{1+2m+\Delta_{(i)}+\Delta_{(j)}-d}\; \bar{\xi}_i^{(0)}(w)\; \bar{\xi}_j^{(m)}(w) \\ \equiv \;\; &\bar{\varphi}_{ij}^{(m)} + \frac{a_i^{IR}\, a_j^{IR}\, \sigma_j^{(m)}\; z^{2+2m+\Delta_{(i)}+\Delta_{(j)}-d}}{2+2m+\Delta_{(i)}+\Delta_{(j)}-d} \bar{F}_{ij}^{(m)}(z) \qquad ; \quad m = 0, 1, \ldots \end{aligned} \tag{A.28}$$

where $\bar{\varphi}_{ij}^{(m)}$ is defined to be the only z-independent part in the large z expansion, and

$$\sigma_{\pm}^{(m)} \equiv \frac{\Gamma(1 \mp \nu_{IR})}{2^{2m}\, m!\, \Gamma(1 \mp \nu_{IR} + m)} \qquad ; \qquad m = 0, 1, \dots \tag{A.29}$$

With them we can calculate ($m = 1, 2, \dots$),

$$\begin{aligned} \varphi_{ij}^{(m)} &\equiv \bar{\varphi}_{ij}^{(m)} + \sum_{k=0}^{m-1} \left(\bar{\varphi}_{i-}^{(k)}\, \varphi_{+j}^{(m-1-k)} - \bar{\varphi}_{i+}^{(k)}\, \varphi_{-j}^{(m-1-k)} \right) & \text{(A.30)} \\ \bar{\xi}_{\pm}^{(m)}(z) &\equiv \frac{1}{\nu_{IR}} \left((\nu_{IR} \mp m)\, \bar{F}_{\mp\pm}^{(m-1)}(z)\, \bar{\xi}_{\pm}^{(0)}(z) \pm m\, \bar{F}_{\pm\pm}^{(m-1)}(z)\, \bar{\xi}_{\mp}^{(0)}(z) \right) & \text{(A.31)} \end{aligned}$$

The general form of $\xi_{\pm}^{(m)}(z)$ for $m = 1, 2, \dots$, results,

$$\begin{aligned} \xi_{\pm}^{(m)}(z) &= a_{\pm}^{IR}\, \sigma_{\pm}^{(m)}\, z^{\Delta_{\mp}^{IR}+2m}\, \bar{\xi}_{\pm}^{(m)}(z) & \text{(A.32)} \\ &+ \sum_{k=0}^{m-1} \left(-a_{+}^{IR}\, \sigma_{+}^{(k)}\, \varphi_{-\pm}^{(m-1-k)}\, z^{\Delta_{-}^{IR}+2k}\, \bar{\xi}_{+}^{(k)}(z) + a_{-}^{IR}\, \sigma_{-}^{(k)}\, \varphi_{+\pm}^{(m-1-k)}\, z^{\Delta_{+}^{IR}+2k}\, \bar{\xi}_{-}^{(k)}(z) \right) \end{aligned}$$

where the ingredients to construct it are iteratively computed as described above.

A.5 The matching procedure.

According to (A.8) and (A.15), in the region

$$z > z_{\infty} \qquad ; \qquad x \equiv qz \ll m_{IR} \tag{A.33}$$

both expansions (A.11) and (A.23) hold and therefore they should coincide *exactly*, i.e.

$$\xi_{\infty}(z;q) \left(1 + \frac{f_0(qz)}{z^{-\Delta_{-}^{IR}}} + \dots \right) = C_{+}(q) \sum_{m \geq 0} q^{2m}\, \xi_{+}^{(m)}(z) + C_{-}(q) \sum_{m \geq 0} q^{2m}\, \xi_{-}^{(m)}(z) \tag{A.34}$$

This equation must be used to compute the unknown coefficients $C^{\pm}(q)$. As we will see shortly, this is not an easy task in general; fortunately the leading order behavior necessary to compute (A.27) is relatively simple to get. To proceed we need the IR behavior of the $\xi_{\pm}^{(m)}(z)$'s. The iterative procedure to get it following the procedure described after (A.20) and (A.21) is developed in the appendix; by plugging (A.32) in (A.23) we get,

$$\begin{aligned} & z^{-\Delta_{-}^{IR}}\, \xi(z;q) = z^{-\Delta_{-}^{IR}}\, r.h.s.\,(A.34) \\ = \;& \left(\left(1 - q^2\, \epsilon_{-+}^{IR}(q)\right) C_{+}(q) - q^2\, \epsilon_{--}^{IR}(q)\, C_{-}(q) \right) a_{+}^{IR} \sum_{m \geq 0} \sigma_{+}^{(m)}\, x^{2m}\, \bar{\xi}_{+}^{(m)} \left(\frac{x}{q} \right) \\ + \;& \left(\left(1 + q^2\, \epsilon_{+-}^{IR}(q)\right) C_{-}(q) + q^2\, \epsilon_{++}^{IR}(q)\, C_{+}(q) \right) \frac{a_{-}^{IR}}{q^{2\nu_{IR}}} \sum_{m \geq 0} \sigma_{-}^{(m)}\, x^{2m+2\nu_{IR}}\, \bar{\xi}_{-}^{(m)} \left(\frac{x}{q} \right) \end{aligned} \tag{A.35}$$

where we have introduced the holomorphic functions,

$$\epsilon^{IR}_{ij}(q) \equiv \sum_{m\geq 0} \varphi^{(m)}_{ij}\; q^{2m} \tag{A.36}$$

On the other hand, by using the series expansion of $\xi_\infty\left(\frac{x}{q};q\right)$ valid for $x<1$ we have,

$$z^{-\Delta^{IR}_-}\,\xi_\infty(z;q) = \sum_{m\geq 0}\left(\sigma^{(m)}_+\; x^{2m} + \gamma\,\sigma^{(m)}_-\; x^{2m+2\nu_{IR}}\right) \qquad ; \qquad \gamma \equiv \frac{\Gamma(-\nu_{IR})}{2^{2\,\nu_{IR}}\,\Gamma(\nu_{IR})} \tag{A.37}$$

Now from (A.34) we have that at fixed $x < minimum(m_{IR},1)$, in the limit $q\to 0$ equations (A.35) and (A.37) should coincide. More specifically, if we introduce $\delta C_\pm(q)$ by,

$$\begin{aligned} C_+(q) &\equiv \frac{1}{D(q)}\left(\frac{1}{a^{IR}_+}\left(1+q^2\,\epsilon^{IR}_{+-}(q)\right) + \frac{\gamma}{a^{IR}_-}\,q^{2+2\nu_{IR}}\,\epsilon^{IR}_{--}(q)\right) + \delta C_+(q)\\ C_-(q) &\equiv \frac{1}{D(q)}\left(-\frac{1}{a^{IR}_+}\,q^2\,\epsilon^{IR}_{++}(q) + \frac{\gamma}{a^{IR}_-}\,q^{2\nu_{IR}}\,\left(1-q^2\,\epsilon^{IR}_{-+}(q)\right)\right) + \delta C_-(q) \end{aligned} \tag{A.38}$$

where,

$$D(q) = 1 + q^2\,\left(\epsilon^{IR}_{+-}(q) - \epsilon^{IR}_{-+}(q)\right) + q^4\,\left(\epsilon^{IR}_{++}(q)\,\epsilon^{IR}_{--}(q) - \epsilon^{IR}_{+-}(q)\,\epsilon^{IR}_{-+}(q)\right) \tag{A.39}$$

then we should get,

$$\begin{aligned} &\lim_{q\to 0}\left\{\sum_{m\geq 0}\sigma^{(m)}_+\,x^{2m}\,\left(\bar{\xi}^{(m)}_+\left(\frac{x}{q}\right)-1\right) + \gamma\,\sum_{m\geq 0}\sigma^{(m)}_-\,x^{2m+2\nu_{IR}}\,\left(\bar{\xi}^{(m)}_-\left(\frac{x}{q}\right)-1\right)\right.\\ &+\;\left(\left(1-q^2\,\epsilon^{IR}_{-+}(q)\right)\,\delta C_+(q) - q^2\,\epsilon^{IR}_{--}(q)\,\delta C_-(q)\right)\,a^{IR}_+\sum_{m\geq 0}\sigma^{(m)}_+\,x^{2m}\,\bar{\xi}^{(m)}_+\left(\frac{x}{q}\right)\\ &+\;\left.\left(\left(1+q^2\,\epsilon^{IR}_{+-}(q)\right)\delta C_-(q) + q^2\,\epsilon^{IR}_{++}(q)\delta C_+(q)\right)\,\frac{a^{IR}_-}{q^{2\nu_{IR}}}\,\sum_{m\geq 0}\sigma^{(m)}_-\,x^{2m+2\nu_{IR}}\,\bar{\xi}^{(m)}_-\,\left(\frac{x}{q}\right)\right\} = 0 \end{aligned} \tag{A.40}$$

While the first line is automatically zero, the second and third lines should be zero separately because they present different power series [9]. From the third line we get,

$$\delta C_-(q) \xrightarrow{q\to 0} -\varphi^{(0)}_{++}\,q^2\,\delta C_+(q) + q^{2\nu_{IR}}\,A(q) \tag{A.41}$$

[9] A subtlety (not present in the case considered in the text) arises if $\bar{\xi}^{(m)}_-(z)$ contains powers of the form $z^{-2\nu_{IR}-2n}$ with $n\in\aleph$; in that case it can be easily showed that the effect is that the coefficients of $\delta C^\pm(q)$ on the second line of (A.40) get modified by holomorphic functions; this fact does not modify the subsequent arguments.

where $A(q) \xrightarrow{q\to 0} 0$. Then the second line of (A.40) yields,

$$\delta C_+(q) \xrightarrow{q\to 0} \varphi_{--}^{(0)}\ q^{2\nu_{IR}+2}\ A(q) \quad \Rightarrow \quad \delta C_-(q) \xrightarrow{q\to 0} q^{2\nu_{IR}}\ A(q) \tag{A.42}$$

Going to (A.38) with (A.42) we get the leading behaviors,

$$C_+(0) = \frac{1}{a_+^{IR}} \qquad ; \qquad C_-(q)|_{q\to 0} = -\frac{\bar{\varphi}_{++}^{(0)}}{a_+^{IR}}\ q^2 \tag{A.43}$$

This yields for the two-point function (A.27) the Goldstone pole,

$$G_2(q) \xrightarrow{q\to 0} \frac{\alpha}{q^2} \tag{A.44}$$

where by using (A.26) and (A.28), i.e.

$$\bar{\varphi}_{++}^{(0)} = \int_{z_i}^{\infty} dw\ w^{1-d}\ \xi_+(w)^2 \tag{A.45}$$

we get for the residue,

$$\alpha = \frac{2\,\nu_{UV}\,(a_+^{UV})^2}{\int_0^\infty dw\ w^{1-d}\ \xi_+(w)^2} \tag{A.46}$$

The result is reassuring in the sense that both contributions in the denominator of (A.27) add to yield a z_i-independent result.

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